JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 97, 229–244 (1983)

Some Generic Properties of Functional-Differential Equations of Neutral Type

MICHAŁ KISIELEWICZ

Institute of Mathematics and Physics, Technical University, Zielona Góra, Poland

Submitted by Kenneth L. Cooke

1. INTRODUCTION

There has recently been a great deal of interest in the field of functional-differential equations of neutral type. These are differential equations in which the present dynamics of the system are influenced by its past behaviour [2]. One of the problems basic to the analysis of such systems is the selection of an appropriate space for solutions. This problem was investigated by Driver [8], Hale and Meyer [9], and Melvin [15, 16].

The main result of the present paper states that all functional-differential equations of neutral type with existence, uniqueness, and continuous dependence on the initial data and the right-hand side of solutions constitute a residual set in some complete metric space.

It is proved furthermore, that nonconvergence of successive approximations of such equations is in any sense a rare case. This property of differential equations is said to be generic.

The study of generic properties of differential equations was started by Orlicz [17], who showed that the subset of all f for which the Cauchy problem for the equation $x' = f(t, x)$ does not have unique solutions is of the first category in the space of all continuous and bounded f with values in the n-dimensional Euclidean space R^n, equipped with a natural metric. This result has been generalized by Alexiewicz and Orlicz [1], Lasota and Yorke [14], Kisielewicz [11–13], Castello [3], Piórek [20], De Blasi and Myjak [4–6], Vidossich [21, 22], and Orlicz and Szufla [18].

Generic properties of functional equations have been considered in [7]. This last paper contains, among others, generic properties of a functional-integral equation

$$y(t) = f\left(t, \int_0^{\alpha(t)} h(t, s, y(s))\, ds, y(\beta(t))\right), \tag{1}$$

0022-247X/83 $3.00

which contains, as a particular case, functional-differential equations of neutral type

$$\dot{x}(t) = f(t, x(\alpha(t)), \dot{x}(\beta(t))). \tag{2}$$

It was proved in [7] that all equations of the form (1) with existence, uniqueness, and continuous dependence on the right-hand side and initial conditions of solutions constitute a residual set in the space of all such equations with continuous right hand sides satisfying a Lipschitz condition with respect to its third variable.

The result of this paper generalizes, among others, results of De Blasi *et al.* [7] concerning functional-differential equation of the form (2).

Let C_α and L_α denote the Banach spaces $C([-r, \alpha], R^n)$ and $L([-r, \alpha], R^n)$ with the usual norms $\|\cdot\|_\alpha$ and $|\cdot|_\alpha$, respectively, where $r \geqslant 0$ and R^n is the n-dimensional Euclidean space with a norm $|\cdot|$. Denote by $\mathscr{L}_{\alpha\beta}(\alpha < \beta)$, the Banach space $L([\alpha, \beta], R^n)$ with the usual norm $|\cdot|_{\alpha\beta}$. If $\alpha = 0$ we will write $\mathscr{L}_\beta$ and $|\cdot|_\beta$ instead of $\mathscr{L}_{0\beta}$ and $|\cdot|_{0\beta}$, respectively. Let us denote by $\mathscr{A}([-r, \alpha], R^n)$ the space of all absolutely continuous functions $x: [-r, \alpha] \to R^n$. It was proved in [19] that $\mathscr{A}([-r, \alpha], R^n)$ together with a metric generated by $\||\cdot|\|_\alpha$ defined by $\||x|\|_\alpha = \|x\|_\alpha + |\dot{x}|_\alpha$, is a complete metric space.

The results of this paper are concerned with functional-differential equations of the form

$$\dot{x}(t) = f(t, x, \dot{x}) \qquad \text{for a.e.} \quad t \in [0, T], \tag{3}$$

with $f: [0, T] \times C_T \times L_T \to R^n$ satisfying the following conditions:

(i) $f(\cdot, x, y) \in \mathscr{L}_T$ for fixed $(x, y) \in C_T \times L_T$;

(ii) the mapping $g: C_T \times L_T \to \mathscr{L}_T$ defined by $g(x, y) = f(\cdot, x, y)$ for $(x, y) \in C_T \times L_T$, is continuous.

We say that f satisfying (i) is locally Lipschitzean if for every $(x, y) \in C_T \times L_T$ there are a neighborhood U_{xy} of (x, y) and a continuous increasing function $K_{xy}: [0, T] \to [0, \infty)$ with $K_{xy}(0) = 0$ such that for every $(x_1, y_1), (x_2, y_2) \in U_{xy}$ and $0 \leqslant \alpha < t \leqslant T$ we have

$$|f(\cdot, x_1, y_1) - f(\cdot, x_2, y_2)|_{\alpha t} \leqslant K_{xy}(t - \alpha)[\|x_1 - x_2\|_t + |y_1 - y_2|_t]. \tag{4}$$

Let us observe that conditions (i) and (ii) proposed above are weaker then the classical Carathéodory conditions. These last conditions are too strong for the functional-differential equations of neutral type. They even rule out difference-differential equations of the form

$$\dot{x}(t) = a x(t) + b x(t-1) + c \dot{x}(t-1).$$

It is not difficult to verify that if $g: [0, T] \times R^{3n} \to R^n$ is such that $g(\cdot, x, y, z) \in \mathscr{L}_T$ for fixed $(x, y, z) \in R^{3n}$ and $|g(t, x_1, y_1, z_1) - g(t, x_2, y_2, z_2)| \leqslant K_1(t)|x_1 - x_2| + K_2(t)|y_1 - y_2| + L|z_1 - z_2|$ for a.e. $t \in [0, T]$, with $K_1, K_2 \in L([0, T], R^+)$ and $L > 0$, then f: $[0, T] \times C_T \times L_T \to R^n$ defined by $f(t, x, y) = g(t, x(t), \dot{x}(\alpha(t)), y(\beta(t)))$ for $t \in [0, T]$ and $(x, y) \in C_T \times L_T$, where α, $\beta \in C([0, T], R)$ are such that $0 \leqslant \alpha(t) \leqslant t$ and $0 \leqslant \beta(t) \leqslant t$, satisfies conditions (i) and (ii).

Let us observe that Eq. (3) contains, as its particular case, a functional-differential equation of the form

$$\dot{x}(t) = f(t, x_t, \dot{x}_t) \qquad \text{for a.e. } t \in [0, T],$$

where for z: $[-r, T] \to R^n$ we put $z_t(s) = z(t + s)$ for fixed $t \in [0, T]$ and $s \in [-r, 0]$.

The results of this paper can be extended to the case when f is defined on an open connected set $D \subset R \times C_T \times L_T$ or takes its values from a separable Banach space.

2. The Metric Space $(\mathscr{F}, \rho)$ and an Approximation Theorem

Let us introduce in the space F of all functions f: $[0, T] \times C_T \times L_T \to R^n$ satisfying conditions (i), (ii), an equivalence relation " $\sim$ " defined by $f_1 \sim f_2$ iff $\sup\{|f_1(\cdot, x, y) - f(\cdot, x, y)|_T \colon (x, y) \in C_T \times L_T\} = 0$.

Let $\mathscr{F}$ denote the space of all equivalence classes of F defined by $\sim$. Notationally, we shall not distinguish between elements of $\mathscr{F}$ and F.

For given $0 \leqslant \alpha < \beta \leqslant T$ and $f_1, f_2 \in \mathscr{F}$, let

$$\rho_{\alpha\beta}(f_1, f_2) = \sup \left\{ \frac{|f_1(\cdot, x, y) - f_2(\cdot, x, y)|_{\alpha\beta}}{1 + |f_1(\cdot, x, y) - f_2(\cdot, x, y)|_{\alpha\beta}} \colon (x, y) \in C_T \times L_T \right\}.$$

In the sequel we will need

Lemma 1. *For every, locally Lipschitzean $f \in \mathscr{F}$, $\lim_{\beta \to \alpha} \rho_{\alpha\beta}(f, 0) = 0$.*

Proof. Suppose $\rho_{\alpha\beta}(f, 0)$ is not converging to zero as $\beta \to \alpha$. Then there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < \rho_{\alpha\beta}(f, 0) \leqslant 1$ for each $0 \leqslant \alpha < \beta \leqslant T$. Hence it follows that for every $0 \leqslant \alpha < \beta \leqslant T$ there exists $(x_0, y_0) \in C_T \times L_T$ such that

$$0 < \frac{\varepsilon_0}{1 - \varepsilon_0} < \int_\alpha^\beta |f(t, x_0, y_0)|\, dt.$$

Let U_0 and K_0: $[0, T] \to [0, \infty)$ be a neighborhood of (x_0, y_0) and a continuous increasing function, respectively, such that (4) is satisfied for (x_1, y_1), $(x_2, y_2) \in U_1$, and $t \in [0, T]$. Therefore, for every $(x_1, y_1) \in U_0$ we have

$$0 < \frac{\varepsilon_0}{1-\varepsilon_0} < K_0(\beta-\alpha)[\|x_1 - x_0\|_T + |y_1 - y_0|_T]$$

$$+ \int_\alpha^\beta |f(t, x_1, y_1)|\, dt.$$

Hence it follows that $\varepsilon_0 = 0$ and the proof is complete.

Define now a metric ρ for $\mathscr{F}$ by setting $\rho(f_1, f_2) = \rho_{0T}(f_1, f_2)$ for $f_1, f_2 \in \mathscr{F}$.

LEMMA 2. *$(\mathscr{F}, \rho)$ is a complete metric space.*

Proof. Let (f_n) be a Cauchy sequence of $\mathscr{F}$. Then for every $k \geqslant 1$ there is $N_k \geqslant 1$ such that

$$\frac{|f_m(\cdot, x, y) - f_n(\cdot, x, y)|_T}{1 + |f_m(\cdot, x, y) - f_n(\cdot, x, y)|_T} < \frac{1}{2^{k+1}}$$

for n, $m \geqslant N_k$ and $(x, y) \in C_T \times L_T$. Hence it follows that

$$|f_m(\cdot, x, y) - f_n(\cdot, x, y)|_T < \frac{1}{2^k}$$

for n, $m \geqslant N_k$, $(x, y) \in C_T \times L_T$, and $k \geqslant 1$. Therefore, $(f_n(\cdot, x, y))$ is a Cauchy sequence of $\mathscr{L}_T$. Then there exists $f(\cdot, x, y) \in \mathscr{L}_T$ such that $|f_n(\cdot, x, y) - f(\cdot, x, y)|_T \to 0$ as $n \to \infty$, uniformly with respect to $(x, y) \in C_T \times L_T$. Hence from continuity of g_n: $C_T \times L_T \ni (x, y) \to f_n(\cdot, x, y) \in \mathscr{L}_T$ it follows that a mapping g: $C_T \times L_T \ni (x, y) \to f(\cdot, x, y) \in \mathscr{L}_T$ is continuous. Then $f \in \mathscr{F}$. We have of course $\rho(f_n, f) \to 0$ as $n \to \infty$ which completes the proof.

Adopting now the procedure of Lasota and Yorke [14] we shall show that every $f \in \mathscr{F}$ can be approximated by locally Lipschitzean elements of this space.

THEOREM 3. *For every $f \in \mathscr{F}$ and $\varepsilon > 0$ there exists $f_\varepsilon \in \mathscr{F}$, locally Lipschitzean and such that $\rho(f_\varepsilon, f) \leqslant \varepsilon$.*

Proof. Denote by $p_t(x)$ and $q_t(y)$, for fixed $t \in [0, T]$ and $(x, y) \in C_T \times L_T$, functions of $[-r, T]$ into R^n defined by

$$\begin{aligned} p_t(x)(s) &= x(s) \qquad \text{for} \quad s \in [-r, t], \\ &= x(t) \qquad \text{for} \quad s \in (t, T], \end{aligned}$$

and

$$q_t(y)(s) = y(s) \qquad \text{for} \quad s \in [-r, t],$$
$$= 0 \qquad \text{for} \quad s \in (t, T].$$

We have $p_t(x) \in C_T$, $q_t(y) \in L_T$, $\| p_t(x)\|_T = \|x\|_t$, $|q_t(y)|_T = |y|_t$. It is not difficult to see that $p_t(x)$ and $q_t(y)$ are linear with respect to x and y, respectively.

Moreover, for every $A \times B \subset C_T \times L_T$ such that $\inf\{\| p_t(x-u)\|_T + |q_t(y-w)|_T: (u, w) \in A \times B\}$ exists for $(x, y) \in C_T \times L_T$ and $t \in [0, T]$, a function $h: [0, T] \times C_T \times L_T \to R$ defined by $h(t, x, y) = \inf\{\| p_t(x-u)\|_T + |q_t(y-w)|_T: (u, w) \in A \times B\}$ for $t \in [0, T]$ and $(x, y) \in C_T \times L_T$ is measurable in $t \in [0, T]$ for fixed $(x, y) \in C_T \times L_T$. It remains true if A is in the space $\mathscr{B}_T$ of all bounded functions of $[-r, T]$ into $\mathbb{R}^n$.

Let $\varepsilon > 0$ be given. By the continuity of a mapping $g: C_T \times L_T \ni (x, y) \to f(\cdot, x, y) \in \mathscr{L}_T$, for every $(x, y) \in C_T \times L_T$ there exists a $\delta(x, y, \varepsilon) > 0$ such that $|f(\cdot, x, y) - f(\cdot, \bar{x}, \bar{y})|_T < \varepsilon$ for each $(\bar{x}, \bar{y}) \in C_T \times L_T$ satisfying $\|x - \bar{x}\|_T + |y - \bar{y}|_T < \delta(x, y, \varepsilon)$.

Let $N(x, y, \varepsilon) = \{(u, v) \in C_T \times L_T: \|x - u\|_T + |y - v|_T < \delta(x, y, \varepsilon)$ for $(x, y) \in C_T \times L_T$. For each $(\bar{x}, \bar{y}) \in N(x, y, \varepsilon)$ and $t \in [0, T]$ we have

$$\| p_t(x - \bar{x})\|_T + |q_t(y - \bar{y})|_T < \delta(x, y, \varepsilon).$$

Furthermore, $C_T \times L_T = \bigcup\{N(x, y, \varepsilon): (x, y) \in C_T \times L_T\}$. By the paracompactness of $C_T \times L_T$ there exists an open locally finite refinement $(Q_j)\ j \in A$ of $\{N(x, y, \varepsilon)\}(x, y) \in C_T \times L_T$. Then for every $j \in A$ there exists $(\tilde{x}, \tilde{y}) \in C_T \times L_T$ such that $Q_j \subset N(\tilde{x}, \tilde{y}, \varepsilon)$.

Define, for fixed $t \in [0, T]$, $(x, y) \in C_T \times L_T$, and $j \in A$,

$$\begin{aligned} &r_j(p_t(x), q_t(y)) \\ &\quad = 0 \qquad \text{for } (x, y) \notin Q_j, \\ &\quad = \inf\{\| p_t(x - \bar{x})\|_T + |q_t(y - \bar{y})|_T: (\bar{x}, \bar{y}) \in \partial\hat{Q}_j^C \times \partial Q_j^L\} \end{aligned}$$

for $(x, y) \in Q_j$, where $\partial\hat{Q}_j^C = \{u \in \mathscr{B}_T: u(t) \in \partial Q_j^C(t)$ for $t \in [-r, T]\}$, $\partial Q_j^C(t) = \{u(t): u \in Q_j^C\}$, Q_j^C and Q_j^L denote projections of Q_j onto C_T and L_T, respectively, and ∂B stands for the boundary of a set B. It is not difficult to see that

$$\begin{aligned} &|r_j(p_t(x_1), q_t(y_1)) - r_j(p_t(x_2), q_t(y_2))| \leqslant \| p_t(x_1 - x_2)\|_T \\ &\quad + |q_t(y_1 - y_2)|_T \leqslant \|x_1 - x_2\|_t + |y_1 - y_2|_t \leqslant \|x_1 - x_2\|_T \\ &\quad + |y_1 - y_2|_T \end{aligned}$$

for (x_1, y_1), $(x_2, y_2) \in C_T \times L_T$, $t \in [0, T]$, and $j \in A$. Hence it follows, in particular, that for each $j \in A$ and $t \in [0, T]$ a mapping $r_j(p_t(\cdot), q_t(\cdot))$ of

$C_T \times L_T$ into R is uniformly continuous on $C_T \times L_T$. Therefore, a mapping $w.: [0, T] \times C_T \times L_T \to R$ defined by $w_t(x, y) = [\sum_{j \in A} r_j(p_t(x), q_t(y))]^{-1}$ for every $t \in [0, T]$ and $(x, y) \in C_T \times L_T$, is measurable in $t \in [0, T]$ and continuous on $C_T \times L_T$ for fixed $t \in [0, T]$. Indeed, by the definition of $(Q_j)_{j \in A}$, for every $(x, y) \in C_T \times L_T$, there are a neighborhood U_{xy} of (x, y) and a set $A_{xy} = \{j_1, ..., j_{n_{xy}}\} \subset A$ such that $Q_j \cap U_{xy} = \varnothing$ for $j \notin A_{xy}$. Therefore $r_j(p_t(\tilde{x}, q_t(\tilde{y})) = 0$ for $t \in [0, T]$, $(\tilde{x}, \tilde{y}) \in U_{xy}$ and $j \notin A_{xy}$. For $j \in A_{xy}$ we have $Q_j \cap U_{xy} \neq \varnothing$. Then $\sum_{j \in A_{xy}} r_j(p_t(\tilde{x}), \quad q_t(\tilde{y})) = \sum_{j \in A} r_j(p_t(\tilde{x}), q_t(\tilde{y})) > 0$ for $t \in [0, T]$ and $(\tilde{x}, \tilde{y}) \in C_T \times L_T$. Therefore, we have $w_t(\tilde{x}, \tilde{y}) = [\sum_{j \in A_{xy}} r_j(p_t(\tilde{x}), q_t(\tilde{y}))]^{-1}$ for $(\tilde{x}, \tilde{y}) \in U_{xy}$. Hence it follows that $w.(\cdot, \cdot)$ is continuous on $C_T \times L_T$ for fixed $t \in [0, T]$ and measurable in $t \in [0, T]$.

We shall show now that for every $(x, y) \in C_T \times L_T$ there are a neighborhood U_{xy} of (x, y) and a number $L_{xy} > 0$ such that

$$|v_j(p_t(x_1), q_t(y_1)) - v_j(p_t(x_2), q_t(y_2))| \leqslant L_{xy}(\|x_1 - x_2\|_t + |y_1 - y_2|_t)$$

for each (x_1, y_1), $(x_2, y_2) \in U_{xy}$, $t \in [0, T]$, and $j \in A$, where $v_j(p_t(x), q_t(y)) = r_j(p_t(x), q_t(y)) \cdot w_t(x, y)$.

Let $t \in [0, T]$ and $j \in A$ be fixed. By the continuity of $w_0(\cdot, \cdot)$ on $C_T \times L_T$, for every $(x, y) \in C_T \times L_T$ there exists a neighborhood, say again U_{xy}, of (x, y) such that $w_0(\tilde{x}, \tilde{y}) < 1 + w_0(x, y)$ for $(\tilde{x}, \tilde{y}) \in U_{xy}$. Suppose U_{xy} is such taken, that

$$w_t(\tilde{x}, \tilde{y}) = \left[\sum_{j \in A_{xy}} r_j(p_t(\tilde{x}), q_t(\tilde{y}))\right]^{-1}$$

and

$$w_t(\tilde{x}, \tilde{y}) < w_0(\tilde{x}, \tilde{y}) < 1 + w_0(x, y)$$

for $(\tilde{x}, \tilde{y}) \in U_{xy}$. For (x_1, y_1), $(x_2, y_2) \in U_{xy}$, we have

$$\begin{aligned}
&|v_j(p_t(x_1), q_t(y_1)) - v_j(p_t(x_2), q_t(y_2))| \\
&\quad \leqslant w_t(x_1, y_1)[|r_j(p_t(x_1), q_t(y_1)) - r_j(p_t(x_2), q_t(y_2))| \\
&\quad\quad + \sum_{j \in A_{xy}} |r_j(p_t(x_1), q_t(y_1)) - r_j(p_t(x_2), q_t(y_2))|] \\
&\quad \leqslant M_{xy}(1 + n_{xy})[\|x_1 - x_2\|_t + |y_1 - y_2|_t],
\end{aligned}$$

where $M_{xy} = 1 + w_0(x, y)$.

Let us define now a desired function f_ε: $[0, T] \times C_T \times L_T \to R^n$ by setting $f_\varepsilon(t, x, y) = g_\varepsilon(t, p_t(x), q_t(y))$, where $g_\varepsilon(s, p_t(x), q_t(y)) = \sum_{j \in A} v_j(p_t(x), q_t(y)) \cdot f(s, x_j, y_j)$ for $0 \leqslant s \leqslant t \leqslant T$ and $(x, y) \in C_T \times L_T$, where $(x_j, y_j) \in Q_j$. Suppose a neighborhood U_{xy} of (x, y), $L_{xy} > 0$, and $A_{xy} = \{j_1, ..., j_{n_{xy}}\} \subset A$ are such that $g_\varepsilon(s, t, \tilde{x}, \tilde{y}) = \sum_{j \in A_{xy}} v_j(p_t(\tilde{x}), q_t(\tilde{y})) \cdot f(s, x_j, y_j)$ and $|v_j(p_t(x_1),$

$q_t(y_1)) - v_j(p_t(x_2), q_t(y_2))| \leqslant L_{xy}[\|x_1 - x_2\|_t + |y_1 - y_2|_t]$ for $0 \leqslant s \leqslant t \leqslant T$ and $(\tilde{x}, \tilde{y})$, (x_1, y_1), $(x_2, y_2) \in U_{xy}$. Hence it follows that g_ε is measurable with respect to $s \in [0, t]$ and measurable with respect to $t \in [0, T]$ and satisfies

$$\begin{aligned} &| g_\varepsilon(s, p_s(x_1), q_s(y_1)) - g_\varepsilon(s, p_s(x_2), q_s(y_2))| \\ &\quad \leqslant k_{xy}(s)[\| p_s(x_1 - x_2)\|_T + |q_s(y_1 - y_2)|_T] \\ &\quad \leqslant k_{xy}(s)[\|x_1 - x_2\|_t + | y_1 - y_2|_t] \end{aligned} \tag{5}$$

for $0 \leqslant s \leqslant t \leqslant T$ and (x_1, y_1), $(x_2, y_2) \in U_{xy}$, where $k_{xy}(s) = n_{xy} L_{xy} \cdot \max\{|f(s, x_j, y_j)| : j \in A_{xy}\}$. Therefore, $f_\varepsilon(\cdot, x, y) \in \mathscr{L}_T$ for fixed $(x, y) \in C_T \times L_T$ and

$$\begin{aligned} &\int_\alpha^t |f_\varepsilon(s, x_1, y_1) - f_\varepsilon(s, x_2, y_2)|\, ds \\ &\quad \leqslant \int_\alpha^t k_{xy}(s)[\|x_1 - x_2\|_t + | y_1 - y_2|_t]\, ds \\ &\quad = K_{xy}(t - \alpha)[\|x_1 - x_2\|_t + | y_1 - y_2|_t] \end{aligned}$$

for (x_1, y_1), $(x_2, y_2) \in U_{xy}$, and $0 \leqslant \alpha \leqslant t \leqslant T$, where $K_{xy}(t - \alpha) = \int_\alpha^t k_{xy}(s)\, ds$. Thus f_ε is locally Lipschitzean. Hence, in particular follows that f_ε satisfies condition (ii) and therefore $f_\varepsilon \in \mathscr{F}$.

We shall show now that $\rho(f_\varepsilon, f) \leqslant \varepsilon$. Let us observe first that for every $s \in [0, t]$, $t \in [0, T]$, and $(x, y) \in C_T \times L_T$ we have $p_t(x)|_{[-r,s]} = p_s(x)|_{[-r,s]}$ and $q_t(y)|_{[-r,s]} = q_s(y)|_{[-r,s]}$. Therefore,

$$\int_0^t | g_\varepsilon(s, p_s(x), q_s(y)) - g_\varepsilon(s, p_t(x), q_t(y))|\, ds = 0$$

for fixed $t \in [0, T]$ and $(x, y) \in C_T \times L_T$. Then, in particular, $f_\varepsilon(s, x, y) = g_\varepsilon(s, p_t(x), q_t(y))$ for $t \in [0, T]$, $(x, y) \in C_T \times L_T$ and a.e. $s \in [0, t]$.

Suppose now, a neighborhood U_{xy} of $(x, y) \in C_T \times L_T$ and a set $A_{xy} = \{j_1, \ldots, j_{n_{xy}}\} \subset A$ are such that $(x, y) \in \bigcap_{j \in A_{xy}} Q_j$ and $(x, y) \notin Q_j$ for $j \in A \backslash A_{xy}$. For fixed $t \in [0, T]$ and $(x, y) \in C_T \times L_T$ we have

$$\begin{aligned} &\int_0^t |f_\varepsilon(s, x, y) - f(s, x, y)|\, ds = \int_0^t | g_\varepsilon(s, p_t(x), q_t(y)) \\ &\quad - f(s, x, y)|\, ds \leqslant \sum_{j \in A_{xy}} v_j(p_t(x), q_t(y)) \int_0^T |f(s, x_j, y_j) \\ &\quad - f(s, x, y)|\, ds \leqslant \varepsilon \cdot \sum_{j \in A_{xy}} v_j(p_t(x), q_t(y)) = \varepsilon. \end{aligned}$$

Hence it follows that $\rho(f_\varepsilon, f) \leqslant \varepsilon$ and the proof is complete.

3. Existence, Uniqueness, and Continuous Dependence of Solutions

Let us consider an initial value problem

$$\begin{aligned} \dot{x}(t) &= f(t, x, \dot{x}) \qquad \text{for a.e. } t \in [0, T], \\ x(t) &= \varphi(t) \qquad \text{for } t \in [-r, 0], \end{aligned} \tag{6}$$

where $f \in \mathscr{F}$ and $\varphi \in \mathscr{A}([-r, 0], R^n)$. By a solution of (6) we mean an absolutely continuous function $x\colon [-r, T] \to R^n$ satisfying (6).

Adopting now the classical method of successive approximations, we will prove that if $f \in \mathscr{F}$ is locally Lipschitzean, then (6) has exactly one solution.

Theorem 4. *Let $f \in \mathscr{F}$ be locally Lipschitzean. Then, for every $\varphi \in \mathscr{A}([-r, 0], R^n)$ there exists exactly one solution of* (6) *on the maximal interval $[-r, \hat{T}_f] \subset [-r, T]$.*

Proof. Let $x_0 \in \mathscr{A}([-r, T], R^n)$ be defined by $x_0(t) = \varphi(t)$ for $t \in [-r, 0]$ and $x_0(t) = \varphi(0)$ for $t \in [0, T]$. Suppose U_0 and $K_0\colon [0, T] \to R$ are a neighborhood of $(x_0, \dot{x}_0)$ and a continuous increasing function, respectively, such that

$$|f(\cdot, x_1, y_1) - f(\cdot, x_2, y_2)|_t \leqslant K_0(t)[\|x_1 - x_2\|_t + | y_1 - y_2 |_t]$$

for (x_1, y_1), $(x_2, y_2) \in U_0$, and $t \in [0, T]$.

Let S_0 be a closed ball of $C_T \times L_T$ with the center $(x_0, \dot{x}_0)$ and a radius $r_0 > 0$ such that $S_0 \subset U_0$. Select $T_0 \in [0, T]$ such that $K_0(T_0) < \frac{1}{2}$, $\rho T_0(f, 0) < 1$, and $(\rho T_0(f, 0))/(1 - \rho_{T_0}(f, 0)) \leqslant r_0/2$. This is possible, because $\rho_t(f, 0) \to 0$ and $K_0(t) \to 0$ as $t \to 0$. Hence it follows that $\int_0^{T_0} |f(t, x, y)| \, dt \leqslant r_0/2$ for each $(x, y) \in C_T \times L_T$.

Let us define a sequence (x_n) of $\mathscr{A}([-r, T], R^n)$ by setting

$$\begin{aligned} x_n(t) &= \varphi(t) \qquad \text{for} \quad t \in [-r, 0], \\ &= \varphi(0) + \int_0^t f(s, x_{n-1}, \dot{x}_{n-1}) \, ds \qquad \text{for } t \in [0, T_0], \\ &= x_n(T_0) \qquad \text{for} \quad t \in (T_0, T], \end{aligned}$$

for $n = 1, 2, \ldots$, where x_0 was defined above. We have $x_n \in \mathscr{A}([-r, T], R^n)$, $\|x_n - x_0\|_T \leqslant r_0/2$, and $|\dot{x}_n - x_0|_T \leqslant r_0/2$ for each $n = 1, 2, \ldots$. Therefore $\|x_n - x_0\|_T \leqslant r_0$ and $(x_n, \dot{x}_n) \in S_0 \subset U_0$ for each $n = 1, 2, \ldots$. Hence from the properties f we get $\|x_{n+1} - x_n\|_T \leqslant 2K_0(T_0) \; \|x_n - x_{n-1}\|_T$ for $n = 1, 2, \ldots$. Then (x_n) is a Cauchy sequence of $\mathscr{A}([-r, T], R^n)$, because $K_0(T_0) < \frac{1}{2}$.

Therefore, there exists a $x^1 \in \mathscr{A}([-r, T], R^n)$, such that $\lim_{n\to\infty} \|x_n - x^1\|_T = 0$. We have of course $(x^1, \dot{x}^1) \in S_0 \subset U_0$. Then

$$\left| x^1(t) - \varphi(0) - \int_0^t f(s, x^1, \dot{x}^1)\, ds \right| \leqslant \|x^1 - x_n\|_T$$

$$+ \int_0^t |f(s, x^1, \dot{x}^1) - f(s, x_n, \dot{x}_n)|\, ds \leqslant \|x_1 - x_n\|_T (1 + K_0(T_0))$$

for $t \in [0, T]$ and $n = 1, 2,\ldots$. Furthermore, $x^1(t) = \varphi(t)$ for $t \in [-r, 0]$. Thus

$$\begin{aligned} x^1(t) &= \varphi(t) && \text{for} \quad t \in [-r, 0], \\ &= \varphi(0) + \int_0^t f(s, x^1, \dot{x}^1)\, ds && \text{for} \quad t \in [0, T_0], \\ &= x_1(T_0) && \text{for} \quad t \in (T_0, T]. \end{aligned} \tag{7}$$

We shall show now that there exists exactly one function $x^1 \in \mathscr{A}([-r, T], R^n)$ satisfying (7). Indeed, suppose $y^1 \in \mathscr{A}([-r, T], R^n)$ satisfies (7) too. Since $\| y_1 - x^1 \|_T \leqslant 2K(T_0) \| y_1 - x^1 \|_T < \| y_1 - x^1 \|_T$, then $\| y_1 - x^1 \|_T = 0$.

Now, in a similar way as above we can define a $T_1 \in (T_0, T]$ and a unique function $x^2 \in \mathscr{A}([-r, T], R^n)$ such that

$$\begin{aligned} x^2(t) &= x^1(t) && \text{for} \quad t \in [-r, T_0], \\ &= x^1(T_0) + \int_{T_0}^t f(s, x^2, \dot{x}^2)\, ds && \text{for} \quad t \in [T_0, T_1], \\ &= x^2(T_1) && \text{for} \quad t \in (T_1, T]. \end{aligned}$$

Hence it follows that

$$\begin{aligned} x^2(t) &= \varphi(t) && \text{for} \quad t \in [-r, 0], \\ &= \varphi(0) + \int_0^t f(s, x^2, \dot{x}^2)\, ds && \text{for} \quad t \in [0, T_1], \\ &= x^2(T_1) && \text{for} \quad t \in (T_1, T]. \end{aligned}$$

Continuing this process we can define a unique function $x \in \mathscr{A}([-r, T], R^n)$ satisfying (6) on the maximal interval $[-r, \hat{T}_f] \subset [-r, T]$. This completes the proof.

Let us denote by $\Lambda(\varphi, f)$ the set of all solutions of (6) corresponding to $(\varphi, f) \in \mathscr{A}([-r, 0], R^n) \times \mathscr{F}$. For simplicity it will be assumed that $\hat{T}_f = T$ for each $f \in \mathscr{F}$.

THEOREM 5. *Let $x \in \Lambda(\varphi, f)$, where f is locally Lipschitzean and suppose $(\varphi_n, f_n) \in \mathscr{A}([-r, 0), R^n) \times \mathscr{F}$ are such that $\Lambda(\varphi_n, f_n) \neq 0$ for each*

$n=1, 2,\ldots,$ and $\|\varphi_n-\varphi\|_0+\rho(f_n,f)\to 0$ as $n\to\infty$. Then $\|x_n-x\|_T\to 0$ as $n\to\infty$, where $x_n\in \Lambda(\varphi_n,f_n)$.

Proof. Let $p_t(x_n)$ and $q_t(\dot x_n)$ be defined, for fixed $t\in[0,T]$ and $n=1,2,\ldots,$ as in the proof of Theorem 3.

Let $x_0=p_0(\varphi)$ and let U_0 and K_0 be a neighborhood of $(x_0,\dot x_0)$ and a continuous increasing function, respectively, such that

$$|f(\cdot,x_1,y_1)-f(\cdot,x_2,y_2)|_t\leqslant K_0(t)[\|x_1-x_2\|_t+|y_1-y_2|_t]$$

for (x_1,y_1), $(x_2,y_2)\in U_0$, and $t\in[0,T]$. Suppose S_0 is a closed ball of center $(x_0,\dot x_0)$ and a radius $r_0>0$ such that $S_0\in U_0$. Select $T_0\in[0,T]$ and $N\geqslant 1$ such that $\|\varphi_n-\varphi\|_0\leqslant r_0/3$, $\rho(f_n,f)<1$,

$$\frac{\rho(f_n,f)}{1-\rho(f_n,f)}\leqslant r_0/6,\qquad \rho_{T_0}(f,0)<1,\frac{\rho_{T_0}(f,0)}{1-\rho_{T_0}(f,0)}\leqslant r_0/6,$$

and $K_0(T_0)<\frac{1}{2}$. This is possible because $\|\varphi_n-\varphi\|_0\to 0$, $\rho(f_n,f)\to 0$, $\rho_t(f,0)\to 0$, and $K_0(t)\to 0$ as $n\to\infty$ and $t\to 0$, respectively. Hence it follows that

$$\int_0^{T_0}|f_n(t,x,y)-f(t,x,y)|\,dt\leqslant\frac{\rho(f_n,f)}{1-\rho(f_n,f)}\leqslant r_0/6$$

and

$$\int_0^{T_0}|f(t,x,y)|\,ds\leqslant\frac{\rho_{T_0}(f,0)}{1-\rho_{T_0}(f,0)}\leqslant r_0/6$$

for $(x,y)\in C_T\times L_T$ and $n\geqslant N$. Then

$$\begin{aligned}\|p_{T_0}(x_n)-x_0\|_T+|q_{T_0}(x_n)-x_0|_T&\leqslant\|\varphi_n-\varphi\|_0\\&+2\left(\int_0^{T_0}|f_n(t,x_n,\dot x_n)-f(t,x_n,\dot x_n)|\,dt\right.\\&\left.+\int_0^{T_0}|f(t,x_n,\dot x_n)|\,dt\right)\leqslant r_0\end{aligned}$$

for $n\geqslant N$. Therefore, $(p_{T_0}(x_n),\ q_{T_0}(\dot x_n))\in S_0\subset U_0$ for $n\geqslant N$. Thus, for n, $m\geqslant N$ we get

$$\begin{aligned}&\|p_{T_0}(x_n)-p_{T_0}(x_m)\|_T+|q_{T_0}(\dot x_n)-q_{T_0}(\dot x_m)|_T\\&\leqslant\frac{1}{1-2K_0(T_0)}\left[\|\varphi_n-\varphi_m\|_0+2\frac{\rho(f_n,f)}{1-\rho(f_n,f)}\right.\\&\left.+2\frac{\rho(f_m,f)}{1-\rho(f_m,f)}\right].\end{aligned}$$

Then $\{(p_{T_0}(x_n),\ q_{T_0}(\dot{x}_n))\}$ is a Cauchy sequence of $\mathscr{A}([-r, T], R^n)$. Therefore, there exists a $x^0 \in \mathscr{A}([-r, T], R^n)$ such that $\|x_n - x^0\|_{T_0} \to 0$ as $n \to \infty$ and

$$x^0(t) = \varphi(t) \qquad \text{for } t \in [-r, 0],$$

$$\dot{x}^0(t) = f(t, x^0, \dot{x}^0) \qquad \text{for a.e. } t \in [0, T_0].$$

Continuing this process, we can define a function $\bar{x} \in \mathscr{A}([-r, T], R^n)$ such that $\bar{x} \in \Lambda(\varphi, f)$ and $\|x_n - \bar{x}\|_T \to 0$ as $n \to \infty$. But $\Lambda(\varphi, f) = \{x\}$. Then $\|x_n - x\|_T \to 0$ as $n \to \infty$, which completes the proof.

4. Generic Properties of Functional-Differential Equations of Neutral Type

We will need in this section the following lemma presented in [10].

Lemma 6. *Suppose (Z, d) is a Baire space and let $A \subset B \subset Z$. If A is a residual subset of (Z, d), then B is a residual subset of (Z, d), too.*

Furthermore, we will use here the following, unpublished result of Lasota.

Lemma 7. *Let (X, d) be a complete metric space and S a dense subset of (X, d). Suppose a function $\chi: X \to [0, \infty)$ is such that $\chi(x_n) \to 0$ for any sequence $(x_n) \subset X$ such that $x_n \to x \in S$. Then the set $\mathscr{X} = \{x \in X: \chi(x) = 0\}$ is a residual subset of (X, d).*

Proof. Let us observe that $X \backslash \mathscr{X} = \bigcup_{k=1}^{\infty} \mathscr{X}_k$, where $\mathscr{X}_k = \{x \in X: \chi(x) \geqslant 1/k\}$. Then it sufficies only to show that $\operatorname{int}(\bar{\mathscr{X}}_k) = \varnothing$ for $k = 1, 2, \ldots$. Suppose, that there exists k such that $\operatorname{int}(\bar{\mathscr{X}}_k) \neq \varnothing$. Then there exists an open ball $B(\hat{x}, \delta)$ with the center $\hat{x}$ and radius $\delta > 0$ such that $B(\hat{x}, \delta) \subset \mathscr{X}_k$. But S is dense in (X, d). Therefore, there exists $x_0 \in B(\hat{x}, \delta) \cap S$ such that $\chi(x_0) = 0$. Furthermore, there exists $\lambda > 0$ such that $B(x_0, \lambda) \subset B(\hat{x}, \delta)$ and such that $\chi(x) < 1/k$ for $x \in B(x_0, \lambda)$. Indeed, suppose that it is not true. Then for $\lambda = 1/n$ there exists $x_n \in B(x_0, 1/n)$ such that $\chi(x_n) \geqslant 1/k$. Of course, $x_n \to x_0 \in S$ as $n \to \infty$, which contradicts the assumption on χ. Then $x_0 \in B(\hat{x}, \delta)$ and $x_0 \in \mathscr{X}_k$. This contradiction completes the proof.

Now we can prove the main results of this paper.

Theorem 8. *Let $\mathscr{X}_1$, $\mathscr{X}_2$, and $\mathscr{X}_3$ denote subsets of $\mathscr{A}([-r, 0], R^n) \times \mathscr{F}$ such that*

(a) *for each $(\varphi, f) \in \mathscr{X}_1$ an initial value problem* (6) *has at most one solution, on the maximal interval $[-r, \hat{T}_f] \subset [-r, T]$,*

(b) *for each* $(\varphi,f)\in\mathscr{X}_2$ *an initial value problem has at least one solution on the maximal interval* $[-r,\hat{T}_f]\subset[-r,T]$,

(c) *for each* $(\varphi,f)\in\mathscr{X}_3$ *a solution* $x(\varphi,f)$ *depends continuously on* (φ,f), *i.e., for every sequence* $\{(\varphi_n,f_n)\}$ *of* $\mathscr{X}_3$ *such that* $\|\varphi_n-\varphi\|_0+\rho(f_n,f)\to 0$ *as* $n\to\infty$ *and* $\Lambda(\varphi_n,f_n)\neq\varnothing$ *we have* $\|x(\varphi_n,f_n)-x(\varphi,f)\|_T\to 0$, *where* $x(\varphi_n,f_n)\in\Lambda(\varphi_n,f_n)$ *for each* $n=1,2,\ldots$.

Then $\mathscr{X}_1\cap\mathscr{X}_2\cap\mathscr{X}_3$ *is a residual subset of* $\mathscr{A}([-r,0],R^n)\times\mathscr{F}$.

Proof. For simplicity it will be assumed that $\hat{T}_f=T$ for each $f\in\mathscr{F}$. Let $X=\mathscr{A}([-r,0],R^n)\times\mathscr{F}$ and let us denote by S the set of all (φ,f) with locally Lipschitzean f. We have $\bar{S}=X$. Define $\chi\colon X\to[0,\infty)$ by setting

$$\chi(\varphi,f)=\lim_{\delta\to 0}\sup\{\|x(\varphi_1,f_1)-x(\varphi_2,f_2)\|_T:(\varphi_1,f_1),$$

$$(\varphi_2,f_2)\in B((\varphi,f),\delta)\},$$

where $(\varphi,f)\in X$, $x(\varphi_i,f_i)\in\Lambda(\varphi_i,f_i)$, $i=1,2$, and $B((\varphi,f),\delta)$ denotes an open ball of X with the center (φ,f) and a radius $\delta>0$.

For each $(\varphi,f)\in X$ and (φ_1,f_1), $(\varphi_2,f_2)\in S\cap B((\varphi,f),\delta)$ we have $\Lambda(\varphi_i,f_i)\neq\varnothing$ for $i=1,2$. Therefore, $\chi(\varphi,f)$ is defined for each $(\varphi,f)\in X$.

Now let us observe that

(i) $(\chi(\varphi,f)=0)\Rightarrow(\Lambda(\varphi,f)$ has at most one point),

(ii) $(\chi(\varphi,f)=0)\Rightarrow(\Lambda(\varphi,f)\neq\varnothing)$,

(iii) $[\chi(\varphi,f)=0$ and $\lim_{n\to\infty}(\|\varphi_n-\varphi\|_0+\rho(f_n,f))=0]\Rightarrow(\lim_{n\to\infty}\chi(\varphi_n,f_n)=0)$,

(iv) $(\chi(\varphi,f)=0)\Rightarrow$ (a solution of (6) depends continuously on (φ,f), and

(v) $(\varphi,f)\in S\Rightarrow\chi(\varphi,f)=0$.

Indeed, suppose $\chi(\varphi,f)=0$ and let $\Lambda(\varphi,f)=\{x,y\}$, where $\|x-y\|_T>0$. Let $\varepsilon_0=\|x-y\|_T$. Since $\lim_{\delta\to 0}\sup\{\|x(\varphi_1,f_1)-x(\varphi_2,f_2)\|_T:\ (\varphi_i,f_i)\in B((\varphi,f),\delta),\ i=1,2\}$, then there exists a $\delta_0>0$ such that $\|x(\varphi_1,f_1)-x(\varphi_2,f_2)\|_T<\varepsilon_0/3$ for each (φ_1,f_1), $(\varphi_2,f_2)\in B((\varphi,f),\delta)$. But $\varepsilon_0=\|x-y\|_T\leqslant\|x-x(\varphi_1,f_1)\|_T+\|x(\varphi_1,f_1)-x(\varphi_2,f_2)\|_T+\|x(\varphi_2,f_2)-y\|_T<\varepsilon_0$. Then $\|x-y\|_T=0$.

Suppose $\chi(\varphi,f)=0$ and let $\{(\varphi,f_n)\}$ be a sequence of S such that $\rho(f_n,f)\to 0$ as $n\to\infty$ and let $x_n=x(\varphi,f_n)\in\Lambda(\varphi,f_n)$ for $n=1,2,\ldots$. For every $\delta>0$ there is an $N\geqslant 1$ such that (φ,f_n), $(\varphi,f_m)\in B((\varphi,f),\delta)$, and $\rho(f_n,f)<1$ for $n,m\geqslant N$. Therefore, we have

$$\|x_n-x_m\|_T\leqslant\sup\{\|x(\varphi_1,f_1)-x(\varphi_2,f_2)\|_T:(\varphi_i,f_i)$$

$$\in B((\varphi,f),\delta),i=1,2\}$$

for n, $m \geqslant N$. From this and $\chi(\varphi,f)=0$ it follows that $\|x_n - x_m\|_T \to 0$ as n, $m \to \infty$. Then, there exists an $x \in \mathscr{A}([-r,T],R^n)$ such that $\|x_n - x\|_T \to 0$ as $n \to \infty$. Since $x(t)=\varphi(t)$ for $t \in [-r,0]$ and

$$\left| x(t) - \varphi(0) - \int_0^t f(s,x,\dot{x})\,ds \right| \leqslant \|x - x_n\|_T + \frac{\rho(f_n,f)}{1-\rho(f_n,f)} + |f(\cdot,x,\dot{x}) - f(\cdot,x_n,\dot{x}_n)|_T$$

for $t \in [0,T]$ and $n \geqslant N$, then $x \in A(\varphi,f)$.

For the proof of (iii), suppose $\chi(\varphi,f)=0$, $\|\varphi_n - \varphi\|_0 + \rho(f_n,f) \to 0$ as $n \to \infty$ and that $\{\chi(\varphi_n,f_n)\}$ is not converging to 0 as $n \to \infty$. Then, there are $\eta > 0$ and a sequence $(\psi_n,g_n) \in X$ such that $\|\psi_n - \varphi\|_0 + \rho(g_n,f) \to 0$ as $n \to \infty$ and so that $\chi(\psi_n,g_n) \geqslant \eta$. Consequently, we can find subsequences $(\psi^i_{n_k}, g^i_{n_k})$ $(i=1,2)$ of X such that

$$\|\psi^i_{n_k} - \psi_n\|_0 + \rho(g^i_{n_k},g_n) \to 0 \quad \text{as } k \to \infty$$

and so that $x^i_{n_k} = x(\psi^i_{n_k}, g^i_{n_k})$ $(i=1,2)$, satisfy $\|x^1_{n_k} - x^2_{n_k}\|_T \geqslant \eta/2$. But $\|\psi^i_{n_k} - \varphi\|_0 + \rho(g^i_{n_k},f) \to 0$ as $k \to \infty$. Hence $\chi(\varphi,f)=0$, similarly as in the proof of (ii), we get $\lim_{k\to\infty} \|x^1_{n_k} - x^2_{n_k}\|_T = \|x^1 - x^2\|_T$, where x^1, $x^2 \in A(\varphi,f)$. But, in virtue of (i), we have $x^1 = x^2$. Therefore $\eta/2 \leqslant \|x^1_{n_k} - x^2_{n_k}\|_T \to 0$ as $k \to \infty$, which contradicts to $\eta > 0$.

Suppose $\chi(\varphi,f)=0$ and let (φ_n,f_n) be a sequence of X such that $\|\varphi_n - \varphi\|_0 + \rho(f_n,f) \to 0$ as $n \to \infty$ and such that $A(\varphi_n,f_n) \neq \varnothing$ for each $n = 1,2,\ldots$. By virtue of (i) and (ii), $\chi(\varphi,f)=0$ implies that there exists $x \in \mathscr{A}([-r,T],R^n)$ such that $A(\varphi,f)=\{x\}$. Using the techniques mentioned above in the proof of (ii), we can show that for every sequence $x_n \in \mathscr{A}([-r,T],R^n)$ such that $x_n \in A(\varphi_n,f_n)$, we have $\|x_n - x\|_T \to 0$ as $n \to \infty$.

Suppose (v) is not true. Then there exists $(\tilde{\varphi},\tilde{f}) \in S$ such that $\chi(\tilde{\varphi},\tilde{f}) \geqslant \eta > 0$. Therefore, for each $n=1,2,\ldots$, there are (φ^1_n,f^1_n), $(\varphi^2_n,f^2_n) \in B((\tilde{\varphi},\tilde{f}),\ 1/n)$ such that $\|x^1_n - x^2_n\|_T \geqslant \eta/2$, where $x^i_n \in A(\varphi^i_n,f^i_n)$ for $i=1,2$, and $n=1,2,\ldots$. Since $\|\varphi^i_n - \tilde{\varphi}\|_0 + \rho(f^i_n,\tilde{f}) \to 0$ for $i=1,2$, as $n \to \infty$ and $(\tilde{\varphi},\tilde{f}) \in S$, then by virtue of Theorem 6, we have $\eta/2 \leqslant \|x^1 - x^2\|_T = 0$, because x^1, $x^2 \in A(\tilde{\varphi},\tilde{f})$ and $A(\tilde{\varphi},\tilde{f})$ has exactly one point. Therefore, (v) holds.

Let $\Omega = \{(\varphi,f) \in X : \chi(\varphi,f)=0\}$. By virtue of (iii) and Lemma 7, Ω is a residual subset of X. But (i), (ii), and (iv) imply that $\Omega \subset \mathscr{X}_1 \cap \mathscr{X}_2 \cap \mathscr{X}_3$. Therefore, by virtue of Lemma 6, this completes the proof.

Let $(\varphi,f) \in X$ and let $(x_n^{(\varphi,f)})$ be a sequence of $\mathscr{A}([-r, T], R^n)$ defined by

$$
\begin{aligned}
x_n^{(\varphi,f)}(t) &= \varphi(t) \qquad \text{for} \quad t \in [-r, 0] \\
&= \varphi(0) + \int_0^t f(s, x_{n-1}^{(\varphi,f)}, \dot{x}_{n-1}^{(\varphi,f)})\, ds \qquad \text{for} \quad t \in [0, T], \qquad (8)
\end{aligned}
$$

where $x_0^{(\varphi,f)}(t) = \varphi(t)$ for $t \in [-r, 0]$ and $x_0^{(\varphi,f)}(t) = \varphi(0)$ for $t \in [0, T]$.

In a similar way as in the proof of Theorem 4, it can be proved that for every $(\varphi,f) \in S$, $(x_n^{(\varphi,f)})$ is converging in $\mathscr{A}([-r, \hat{T}_f], R^n)$.

Let us denote by $\mathscr{F}_\alpha$ a closed ball of $\mathscr{F}$ with the center 0 and a radius $\alpha \in (0, 1)$ and let $X_\alpha = \mathscr{A}([-r, 0], R^n) \times \mathscr{F}_\alpha$. It is not difficult to verify that the set S_α of all $(\varphi,f) \in X_\alpha$ with locally Lipschitzean f, is a dense subset of X_α.

We shall show now that nonconvergence of a sequence $(x_n^{(\varphi,f)})$ is in any sense a rare case if $(\varphi,f) \in X_\alpha$.

THEOREM 9. *The set $\mathscr{X}$ of all $(\varphi,f) \in X_\alpha$ for which a sequence $(x_n^{(\varphi,f)})$ defined by* (8) *is converging in $\mathscr{A}([-r, T], R^n)$ is a residual subset of X_α for each $\alpha \in (0, 1)$.*

Proof. Let $\chi: x_\alpha \to [0, \infty)$ be defined by $\chi(\varphi,f) = \lim_{m\to\infty} \operatorname{diam} E_m^{(\varphi,f)}$, where $E_m^{(\varphi,f)} = \{x_m^{(\varphi,f)}, x_{m+1}^{(\varphi,f)},...\}$ and $\operatorname{diam} E_m^{(\varphi,f)}$ denotes the diameter of $E_m^{(\varphi,f)}$. We have $E_{m+1}^{(\varphi,f)} \subset E_m^{(\varphi,f)}$ and then $\operatorname{diam} E_{m+1}^{(\varphi,f)} \leqslant \operatorname{diam} E_m^{(\varphi,f)}$ for each $m = 1, 2,....$ Since $0 \leqslant \operatorname{diam} E_1^{(\varphi,f)} \leqslant (4\rho(f, 0))/(1 - \rho(f, 0))$ for $(\varphi,f) \in X_\alpha$, then $\lim_{m\to\infty} \operatorname{diam} E_m^{(\varphi,f)}$ exists for each $(\varphi,f) \in X_\alpha$. We have of course $\chi(\varphi,f) = 0$ iff $(x_m^{(\varphi,f)})$ is converging in $\mathscr{A}([-r, T], R^n)$.

We shall show now that $\chi(\varphi_n,f_n) \to 0$ for every sequence (φ_n,f_n) of X_α such that $\|\varphi_n - \varphi\|_0 + \rho(f_n,f) \to 0$ as $n \to \infty$, where $(\varphi,f) \in S_\alpha$.

Let us observe first that $\|x_m^{(\varphi_n,f_n)} - x_m^{(\varphi,f)}\|_T \to 0$ as $n, m \to \infty$. Indeed, let $x_0(t) = \varphi(t)$ for $t \in [-r, 0]$ and $x_0(t) = \varphi(0)$ $t \in [0, T]$. Select a neighborhood U_0 of $(x_0, \dot{x}_0)$ and a continuous increasing function K_0: $[0, T] \to [0, \infty)$ such that $|f(\cdot, x_1, y_1) - f(\cdot, x_2, y_2)|_t \leqslant K_0(t)[|||x_1 - x_2|||_t + |y_1 - y_2|_t]$ for (x_1, y_1), $(x_2, y_2) \in U_0$, and $t \in [0, T]$. Let B_0 be a closed ball of $C_T \times L_T$ with the center $(x_0, \dot{x}_0)$ and a radius $r_0 > 0$ such that $B_0 \subset U_0$. Similarly as in the proof of Theorem 6, we can select $T_0 \in [0, T]$ and M, $N \geqslant 1$ such that $(p_{T_0}(x_m^{(\varphi,f)}), q_{T_0}(\dot{x}_m^{(\varphi,f)}))$, $(p_{T_0}(x_m^{(\varphi_n,f_n)}), q_{T_0}(\dot{x}_m^{(\varphi_n,f_n)})) \in B_0 \subset U_0$ for $n \geqslant N$ and $m \geqslant M$.

Therefore,

$$
\begin{aligned}
\| p_{T_0}(x_m^{(\varphi_n,f_n)}) - p_{T_0}(x_m^{(\varphi,f)})\|_T &+ |q_{T_0}(\dot{x}_m^{(\varphi_n,f_n)}) \\
- q_{T_0}(\dot{x}_m^{(\varphi,f)})|_T &\leqslant \frac{1}{1 - 2K_0(T_0)} \\
&\times \left[\|\varphi_n - \varphi\|_0 + 2\frac{\rho(f_n,f)}{1 - \rho(f_n,f)} \right]
\end{aligned}
$$

for $n \geqslant N$ and $m \geqslant M$, which proves that $\| p_{T_0}(x_m^{(\varphi_n, f_n)}) - p_{T_0}(x_m^{(\varphi, f)}) \|_T + \| q_{T_0}(\dot{x}_m^{(\varphi_n, f_n)}) - q_{T_0}(\dot{x}_m^{(\varphi, f)}) \|_T \to 0$ as $n, m \to \infty$. Continuing this process we can easily get $\| x_m^{(\varphi_n, f_n)} - x_m^{(\varphi, f)} \|_T \to 0$ as $n, m \to \infty$.

Suppose now that (φ_n, f_n) is a sequence of X_α converging to $(\varphi, f) \in S_\alpha$ such that $\chi(\varphi_n, f_n)$ is not converging to zero. Then there are $\eta > 0$ and a subsequence (φ_k, f_k) of (φ_n, f_n) such that $\| \varphi_k - \varphi \|_0 + \rho(f_k, f) \to 0$ as $k \to \infty$ and $\chi(\varphi_k, f_k) \geqslant \eta$ for each $k \geqslant 1$. Hence, in particular follows that diam $E_m^{(\varphi_k, f_k)} \geqslant \eta$ for k, $m \geqslant 1$. But a sequence $(x_m^{(\varphi_k, f_k)})$ is converging in $\mathscr{A}([-r, T], R^n)$ to $x_m^{(\varphi, f)}$. Therefore, for n, $m \geqslant 1$ and sufficiently large k, say $k \geqslant N$, we have

$$\| x_m^{(\varphi_k, f_k)} - x_n^{(\varphi_k, f_k)} \|_T \leqslant \frac{1}{k} + \| x_m^{(\varphi, f)} - x_n^{(\varphi, f)} \|_T.$$

Thus

$$\eta \leqslant \operatorname{diam} E_m^{(\varphi_k, f_k)} \leqslant \frac{1}{k} + \operatorname{diam} E_m^{(\varphi, f)} = \frac{1}{k}$$

for $k \geqslant N$, because $(\varphi, f) \in S_\alpha$. This contradicts $\eta > 0$.

Now, similarly as in the proof of Theorem 8, we can see that a set $\Omega_\alpha = \{(\varphi, f) \in X_\alpha : \chi(\varphi, f) = 0\}$ is a residual subset of X_α. Therefore, $\mathscr{X}$ is a residual subset of X_α, too. The proof is complete.

Acknowledgments

The author is indebted to the referee of this paper for valuable remarks concerning the Carathéodory conditions in the theory of functional-differential equations of neutral type.

References

1. A. Alexiewicz and W. Orlicz, Some remarks on the existence and uniqueness of solutions of hyperbolic equation $z''_{xy} = f(x, y, z, z'_x, z'_y)$, *Studia Math.* **15** (1965), 201–215.
2. R. Bellman and K. L. Cooke, "Diferential-Difference Equations," Academic Press, New York, 1963.
3. T. Costello, Generic properties of differential equations *SIAM J. Math. Anal.* **4** (1973), 245–249.
4. F. S. De Blasi and J. Myjak, Generic properties of hyperbolic partial differential equations, *J. London Math. Soc.* (*2*) **15** (1977), 113–118.
5. F. S. De Blasi and J. Myjak, Generic properties of differential equations in Banach space, *Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys.* (*14*) **26** (1978), 287–292.
6. F. S. De Blasi and J. Myjak, Sur la convergence des approximations successives pour les contractions non lineaires dans un espace de Banach, *C. R. Acad. Sci. Paris sér. A* **283** (1976), 185–187.

7. F. S. De Blasi, J. Myjak, and M. Kwapisz, Generic properties of functional equations, *Nonlinear Anal. Theoret. Appl.* **2** (1978), 239–249.
8. R. D. Driver, Existence and continuous dependence of solutions of a neutral functional-differential equation, *Arch. Rational Mech. Anal.* **19** (1965), 149–166.
9. J. K. Hale and K. R. Meyer, A class of functional equations of neutral type, *Mem. Amer. Math. Soc.* **76** (1967).
10. R. C. Haworth and R. A. Moccy, Baire space, *Dissertationes Math.* **141** (1977).
11. M. Kisielewicz, The Orlicz type theorem for differential integral equations with a lagging argument, *Ann. Polon. Math.* **31** (1975), 67–71.
12. M. Kisielewicz, Description of a class of hereditary differential-integral equations with non-converging successive approximations, *Funkcial. Ekvac.* **19** (1976), 295–300.
13. Kisielewicz, On the non-convergence of successive approximations in the theory of the equations $z''_{xy} = f(x, y, z'_x, z'_y)$, *Ann. Polon. Math.* **35** (1979), 85–93.
14. A. Lasota and J. A. Yorke, The generic property of existence of solutions of differential equations in Banach space, *J. Differential Equations* **13** (1973), 1–12.
15. W. R. Melvin, Topologies for nonlinear functional-differential equations, *J. Differential Equations* **13** (1973), 24–31.
16. W. R. Melvin, A class of neutral functional-differential equations, *J. Differential Equations* **12** (1972), 524–534.
17. W. Orlicz, Zur Theorie der Differentialgleichung $y' = f(x, y)$, *Bull. Acad. Polon. Sci. Sér. A* (1932), 221–228.
18. W. Orlicz and S. Szufla, On the convergence of successive approximations for nonlinear equations in functional spaces, *Bull. Acad. Polon. Sci. Sér. A* 27 (1979), 153–156.
19. N. Parhi and P. C. Das, On a functional-differential equations of neutral type, *J. Differential Equations* 35 (1971), 67–82.
20. J. Piórek, On integral equations in a Banach space, *Ann. Polon. Math.*, in press.
21. G. Vidossich, Existence and uniqueness of fixed points of nonlinear operators as a generic property, *Bol. Soc. Brasil. Math.* **5** (1974), 17–29.
22. G. Vidossich, Most of the successive approximations do convergence, *J. Math. Anal. Appl.* **45** (1974), 127–131.